CHAPTER 11

AC POWER ANALYSIS

An engineer is an unordinary person who can do for one dollar what any ordinary person can do for two dollars.

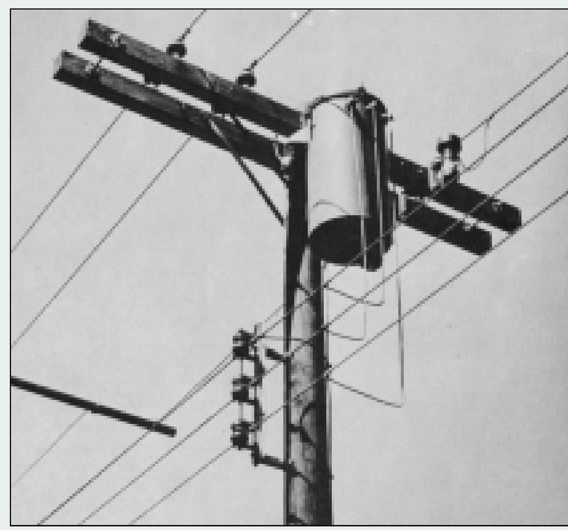
—Anonymous

Enhancing Your Career

Career in Power Systems The discovery of the principle of an ac generator by Michael Faraday in 1831 was a major breakthrough in engineering; it provided a convenient way of generating the electric power that is needed in every electronic, electrical, or electromechanical device we use now.

Electric power is obtained by converting energy from sources such as fossil fuels (gas, oil, and coal), nuclear fuel (uranium), hydro energy (water falling through a head), geothermal energy (hot water, steam), wind energy, tidal energy, and biomass energy (wastes). These various ways of generating electric power are studied in detail in the field of power engineering, which has become an indispensable subdiscipline of electrical engineering. An electrical engineer should be familiar with the analysis, generation, transmission, distribution, and cost of electric power.

The electric power industry is a very large employer of electrical engineers. The industry includes thousands of electric utility systems ranging from large, interconnected systems serving large regional areas to small power companies serving individual communities or factories. Due to the complexity of the power industry, there are numerous electrical engineering jobs in different areas of the industry: power plant (generation), transmission and distribution, maintenance, research, data acquisition and flow control, and management. Since electric power is used everywhere, electric utility companies are everywhere, offering exciting training and steady employment for men and women in thousands of communities throughout the world.



A pole-type transformer with a low-voltage, three-wire distribution system. Source: W. N. Alerich, Electricity, 3rd ed. Albany, NY: Delmar Publishers, 1981, p. 152. (Courtesy of General Electric.)

11.1 INTRODUCTION

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. Our major concern in this chapter is power analysis.

Power analysis is of paramount importance. Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another. Also, every industrial and household electrical device—every fan, motor, lamp, pressing iron, TV, personal computer—has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance. The most common form of electric power is 50- or 60-Hz ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer.

We will begin by defining and deriving *instantaneous power* and *average power*. We will then introduce other power concepts. As practical applications of these concepts, we will discuss how power is measured and reconsider how electric utility companies charge their customers.

Electronic Testing Tutorials

11.2 INSTANTANEOUS AND AVERAGE POWER

As mentioned in Chapter 2, the *instantaneous power* $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it. Assuming the passive sign convention,

$$\boxed{p(t) = v(t)i(t)} \tag{11.1}$$

The instantaneous power is the power at any instant of time. It is the rate at which an element absorbs energy.

We can also think of the instantaneous power as the power absorbed by the element at a specific instant of time. Instantaneous quantities are denoted by lowercase letters.

Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as shown in Fig. 11.1. Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{11.2a}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{11.2b}$$

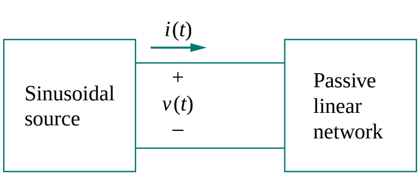


Figure 11.1 Sinusoidal source and passive linear circuit.

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \tag{11.3}$$

We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{11.4}$$

and express Eq. (11.3) as

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i) \tag{11.5}$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.

A sketch of $p(t)$ in Eq. (11.5) is shown in Fig. 11.2, where $T = 2\pi/\omega$ is the period of voltage or current. We observe that $p(t)$ is periodic, $p(t) = p(t + T_0)$, and has a period of $T_0 = T/2$, since its frequency is twice that of voltage or current. We also observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle. When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

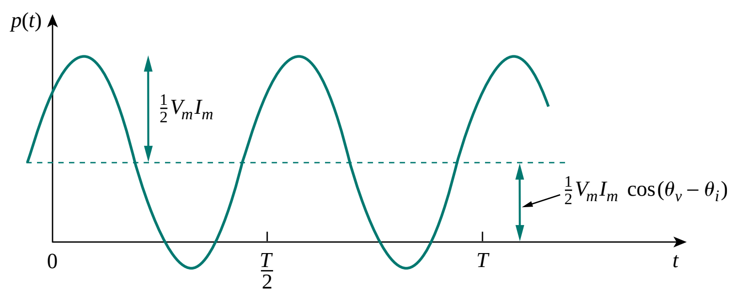


Figure 11.2 The instantaneous power $p(t)$ entering a circuit.

The instantaneous power changes with time and is therefore difficult to measure. The *average* power is more convenient to measure. In fact, the wattmeter, the instrument for measuring power, responds to average power.

The average power is the average of the instantaneous power over one period.

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt \tag{11.6}$$

Although Eq. (11.6) shows the averaging done over T, we would get the same result if we performed the integration over the actual period of $p(t)$ which is $T_0 = T/2$.

Substituting $p(t)$ in Eq. (11.5) into Eq. (11.6) gives

$$\begin{aligned} P &= \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt \\ &\quad + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt \end{aligned}$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T} \int_0^T dt + \frac{1}{2} V_m I_m \frac{1}{T} \int_0^T \cos(2\omega t + \theta_v + \theta_i)\, dt \tag{11.7}$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle. Thus, the second term in Eq. (11.7) vanishes and the average power becomes

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \tag{11.8}$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and current.

Note that $p(t)$ is time-varying while P does not depend on time. To find the instantaneous power, we must necessarily have $v(t)$ and $i(t)$ in the time domain. But we can find the average power when voltage and current are expressed in the time domain, as in Eq. (11.2), or when they are expressed in the frequency domain. The phasor forms of $v(t)$ and $i(t)$ in Eq. (11.2) are $\mathbf{V} = V_m \underline{/\theta_v}$ and $\mathbf{I} = I_m \underline{/\theta_i}$, respectively. P is calculated using Eq. (11.8) or using phasors $\mathbf{V}$ and $\mathbf{I}$. To use phasors, we notice that

$$\begin{aligned} \frac{1}{2}\mathbf{V}\mathbf{I}^* &= \frac{1}{2} V_m I_m \underline{/\theta_v - \theta_i} \\ &= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)] \end{aligned} \tag{11.9}$$

We recognize the real part of this expression as the average power P according to Eq. (11.8). Thus,

$$\boxed{P = \frac{1}{2} \operatorname{Re}\left[\mathbf{V}\mathbf{I}^*\right] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)} \tag{11.10}$$

Consider two special cases of Eq. (11.10). When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R, and

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R \tag{11.11}$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. Equation (11.11) shows that a purely resistive circuit absorbs power at all times. When $\theta_v - \theta_i = \pm 90^\circ$, we have a purely reactive circuit, and

$$P = \frac{1}{2} V_m I_m \cos 90^\circ = 0 \tag{11.12}$$

showing that a purely reactive circuit absorbs no average power. In summary,

> A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

EXAMPLE 11.1

Given that

$$v(t) = 120\cos(377t + 45^\circ)\text{ V} \quad \text{and} \quad i(t) = 10\cos(377t - 10^\circ)\text{ A}$$

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\text{ W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

$$= 600\cos 55^\circ = 344.2\text{ W}$$

which is the constant part of $p(t)$ above.

PRACTICE PROBLEM 11.1

Calculate the instantaneous power and average power absorbed by the passive linear network of Fig. 11.1 if

$$v(t) = 80\cos(10t + 20^\circ)\text{ V} \quad \text{and} \quad i(t) = 15\sin(10t + 60^\circ)\text{ A}$$

Answer: $385.7 + 600\cos(20t - 10^\circ)$ W, 385.7 W.

EXAMPLE 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\underline{/0^\circ}$ is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\underline{/0^\circ}}{30 - j70} = \frac{120\underline{/0^\circ}}{76.16\underline{/-66.8^\circ}} = 1.576\underline{/66.8^\circ} \text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24 \text{ W}$$

PRACTICE PROBLEM 11.2

A current $\mathbf{I} = 10\underline{/30^\circ}$ flows through an impedance $\mathbf{Z} = 20\underline{/-22^\circ}$ Ω. Find the average power delivered to the impedance.

Answer: 927.2 W.

EXAMPLE 11.3

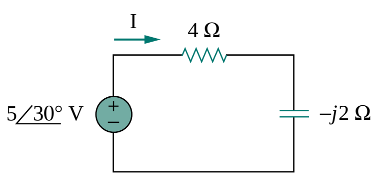


Figure 11.3 For Example 11.3.

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

Solution:

The current $\mathbf{I}$ is given by

$$\mathbf{I} = \frac{5\underline{/30^\circ}}{4 - j2} = \frac{5\underline{/30^\circ}}{4.472\underline{/-26.57^\circ}} = 1.118\underline{/56.57^\circ} \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I} = \mathbf{I}_R = 1.118\underline{/56.57^\circ} \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\underline{/56.57^\circ} \text{ V}$$

The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

PRACTICE PROBLEM 11.3

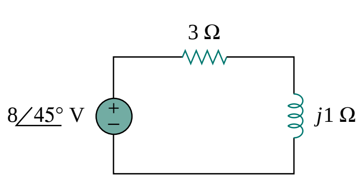


Figure 11.4 For Practice Prob. 11.3.

In the circuit of Fig. 11.4, calculate the average power absorbed by the resistor and inductor. Find the average power supplied by the voltage source.

Answer: 9.6 W, 0 W, 9.6 W.

EXAMPLE 11.4

Determine the power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).

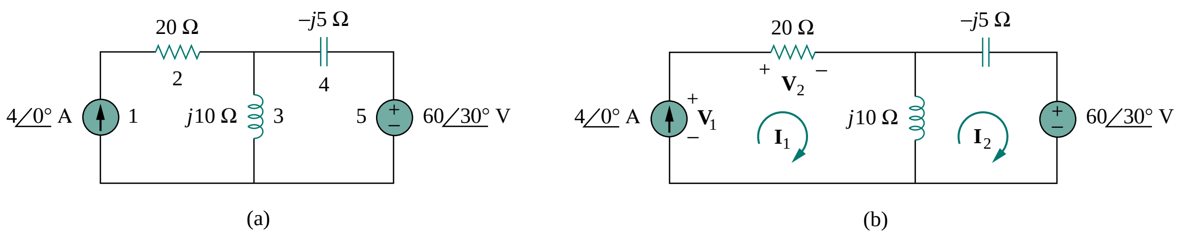


Figure 11.5 For Example 11.4.

Solution:

We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\underline{/30^\circ} = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

or

$$j5\mathbf{I}_2 = -60\underline{/30^\circ} + j40 \qquad \Longrightarrow \qquad \mathbf{I}_2 = -12\underline{/-60^\circ} + 8$$
$$= 10.58\underline{/79.1^\circ} \text{ A}$$

For the voltage source, the current flowing from it is $\mathbf{I}_2 = 10.58\underline{/79.1^\circ}$ A and the voltage across it is $60\underline{/30^\circ}$ V, so that the average power is

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30^\circ - 79.1^\circ) = 207.8 \text{ W}$$

Following the passive sign convention (see Fig. 1.8), this average power is absorbed by the source, in view of the direction of $\mathbf{I}_2$ and the polarity of the voltage source. That is, the circuit is delivering average power to the voltage source.

For the current source, the current through it is $\mathbf{I}_1 = 4\underline{/0^\circ}$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\underline{/6.21^\circ} \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

It is negative according to the passive sign convention, meaning that the current source is supplying power to the circuit.

For the resistor, the current through it is $\mathbf{I}_1 = 4\underline{/0^\circ}$ and the voltage across it is $20\mathbf{I}_1 = 80\underline{/0^\circ}$, so that the power absorbed by the resistor is

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor, the current through it is $\mathbf{I}_2 = 10.58\angle 79.1^\circ$ and the voltage across it is $-j5\mathbf{I}_2 = (5\angle -90^\circ)(10.58\angle 79.1^\circ) = 52.9\angle 79.1^\circ - 90^\circ$. The average power absorbed by the capacitor is

$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90^\circ) = 0$$

For the inductor, the current through it is $\mathbf{I}_1 - \mathbf{I}_2 = 2 - j10.39 = 10.58\angle -79.1^\circ$. The voltage across it is $j10(\mathbf{I}_1 - \mathbf{I}_2) = 105.8\angle -79.1^\circ + 90^\circ$. Hence, the average power absorbed by the inductor is

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90^\circ = 0$$

Notice that the inductor and the capacitor absorb zero average power and that the total power supplied by the current source equals the power absorbed by the resistor and the voltage source, or

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

indicating that power is conserved.

PRACTICE PROBLEM 11.4

Calculate the average power absorbed by each of the five elements in the circuit of Fig. 11.6.

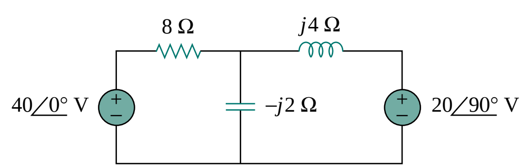


Figure 11.6 For Practice Prob. 11.4.

Answer: 40-V Voltage source: -100 W; resistor: 100 W; others: 0 W.

11.3 MAXIMUM AVERAGE POWER TRANSFER

In Section 4.8 we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load R_L. Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $R_L = R_{\text{Th}}$. We now extend that result to ac circuits.

Consider the circuit in Fig. 11.7, where an ac circuit is connected to a load $\mathbf{Z}_L$ and is represented by its Thevenin equivalent. The load is usually represented by an impedance, which may model an electric motor, an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z}_{\text{Th}}$ and the load impedance $\mathbf{Z}_L$ are

$$\mathbf{Z}_{\text{Th}} = R_{\text{Th}} + jX_{\text{Th}} \tag{11.13a}$$

$$\mathbf{Z}_L = R_L + jX_L \tag{11.13b}$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\text{Th}}}{(R_{\text{Th}} + jX_{\text{Th}}) + (R_L + jX_L)} \tag{11.14}$$

From Eq. (11.11), the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{\text{Th}}|^2 R_L/2}{(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2} \tag{11.15}$$

Our objective is to adjust the load parameters R_L and X_L so that P is maximum. To do this we set $\partial P/\partial R_L$ and $\partial P/\partial X_L$ equal to zero. From Eq. (11.15), we obtain

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\text{Th}}|^2 R_L (X_{\text{Th}} + X_L)}{[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{11.16a}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{\text{Th}}|^2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2 - 2R_L(R_{\text{Th}} + R_L)]}{2[(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2]^2} \tag{11.16b}$$

Setting $\partial P/\partial X_L$ to zero gives

$$X_L = -X_{\text{Th}} \tag{11.17}$$

and setting $\partial P/\partial R_L$ to zero results in

$$R_L = \sqrt{R_{\text{Th}}^2 + (X_{\text{Th}} + X_L)^2} \tag{11.18}$$

Combining Eqs. (11.17) and (11.18) leads to the conclusion that for maximum average power transfer, $\mathbf{Z}_L$ must be selected so that $X_L = -X_{\text{Th}}$ and $R_L = R_{\text{Th}}$, i.e.,

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*} \tag{11.19}$$

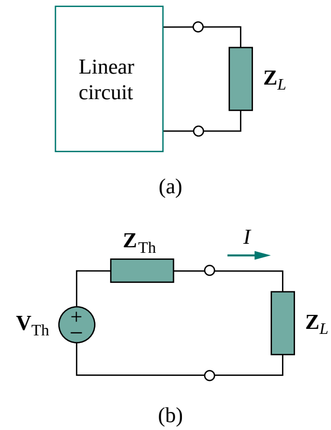


Figure 11.7 Finding the maximum average power transfer: (a) circuit with a load, (b) the Thevenin equivalent.

> For maximum average power transfer, the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{\text{Th}}$.

When $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^*$, we say that the load is matched to the source.

This result is known as the *maximum average power transfer theorem* for the sinusoidal steady state. Setting $R_L = R_{\text{Th}}$ and $X_L = -X_{\text{Th}}$ in Eq. (11.15) gives us the maximum average power as

$$\boxed{P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}} \tag{11.20}$$

In a situation in which the load is purely real, the condition for maximum power transfer is obtained from Eq. (11.18) by setting $X_L = 0$; that is,

$$R_L = \sqrt{R_{\text{Th}}^2 + X_{\text{Th}}^2} = |\mathbf{Z}_{\text{Th}}| \tag{11.21}$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

EXAMPLE 11.5

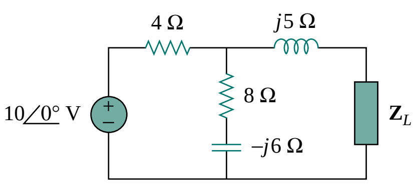


Figure 11.8 For Example 11.5.

Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?

Solution:

First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{\text{Th}}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{\text{Th}} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

To find $\mathbf{V}_{\text{Th}}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\ \text{V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = 2.933 - j4.467\ \Omega$$

According to Eq. (11.20), the maximum average power is

$$P_{\text{max}} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \text{W}$$

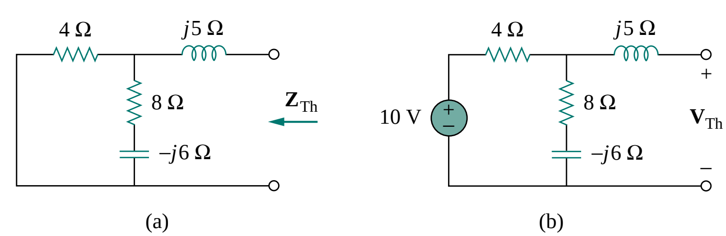


Figure 11.9 Finding the Thevenin equivalent of the circuit in Fig. 11.8.

PRACTICE PROBLEM 11.5

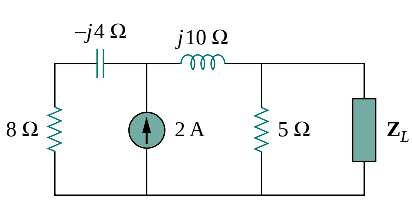


Figure 11.10 For Practice Prob. 11.5.

For the circuit shown in Fig. 11.10, find the load impedance $\mathbf{Z}_L$ that absorbs the maximum average power. Calculate that maximum average power.

Answer: $3.415 - j0.7317\ \Omega$, 1.429 W.

EXAMPLE 11.6

In the circuit in Fig. 11.11, find the value of R_L that will absorb the maximum average power. Calculate that power.

Solution:

We first find the Thevenin equivalent at the terminals of R_L.

$$\mathbf{Z}_{\text{Th}} = (40 - j30) \parallel j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{j20}{j20 + 40 - j30}(150\underline{/30^\circ}) = 72.76\underline{/134^\circ}\ \text{V}$$

The value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{\text{Th}}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + R_L} = \frac{72.76\underline{/134^\circ}}{33.39 + j22.35} = 1.8\underline{/100.2^\circ}\ \text{A}$$

The maximum average power absorbed by R_L is

$$P_{\max} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}(1.8)^2(24.25) = 39.29\ \text{W}$$

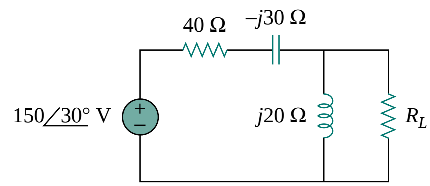


Figure 11.11 For Example 11.6.

PRACTICE PROBLEM 11.6

In Fig. 11.12, the resistor R_L is adjusted until it absorbs the maximum average power. Calculate R_L and the maximum average power absorbed by it.

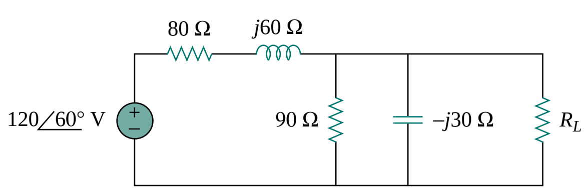


Figure 11.12 For Practice Prob. 11.6.

Answer: 30 Ω, 9.883 W.

11.4 EFFECTIVE OR RMS VALUE

The idea of *effective value* arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.

> The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

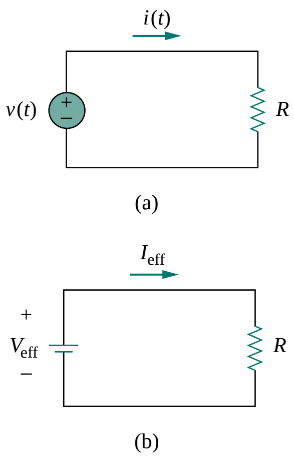


Figure 11.13 Finding the effective current: (a) ac circuit, (b) dc circuit.

In Fig. 11.13, the circuit in (a) is ac while that of (b) is dc. Our objective is to find I_{eff} that will transfer the same power to resistor R as the sinusoid i. The average power absorbed by the resistor in the ac circuit is

$$P = \frac{1}{T}\int_0^T i^2 R\,dt = \frac{R}{T}\int_0^T i^2\,dt \tag{11.22}$$

while the power absorbed by the resistor in the dc circuit is

$$P = I_{\text{eff}}^2 R \tag{11.23}$$

Equating the expressions in Eqs. (11.22) and (11.23) and solving for I_{eff}, we obtain

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} \tag{11.24}$$

The effective value of the voltage is found in the same way as current; that is,

$$V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2\,dt} \tag{11.25}$$

This indicates that the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal. Thus, the effective value is often known as the *root-mean-square* value, or *rms* value for short; and we write

$$I_{\text{eff}} = I_{\text{rms}}, \qquad V_{\text{eff}} = V_{\text{rms}} \tag{11.26}$$

For any periodic function $x(t)$ in general, the rms value is given by

$$\boxed{X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}} \tag{11.27}$$

The effective value of a periodic signal is its root mean square (rms) value.

Equation 11.27 states that to find the rms value of $x(t)$, we first find its *square* x^2 and then find the *mean* of that, or

$$\frac{1}{T}\int_0^T x^2\,dt$$

and then the square *root* ($\sqrt{}$) of that mean. The rms value of a constant is the constant itself. For the sinusoid $i(t) = I_m \cos\omega t$, the effective or rms value is

$$\begin{aligned} I_{\text{rms}} &= \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t\,dt} \\ &= \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1+\cos 2\omega t)\,dt} = \frac{I_m}{\sqrt{2}} \end{aligned} \tag{11.28}$$

Similarly, for $v(t) = V_m \cos \omega t$,

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{11.29}$$

Keep in mind that Eqs. (11.28) and (11.29) are only valid for sinusoidal signals.

The average power in Eq. (11.8) can be written in terms of the rms values.

$$\begin{aligned} P &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \end{aligned} \tag{11.30}$$

Similarly, the average power absorbed by a resistor R in Eq. (11.11) can be written as

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \tag{11.31}$$

When a sinusoidal voltage or current is specified, it is often in terms of its maximum (or peak) value or its rms value, since its average value is zero. The power industries specify phasor magnitudes in terms of their rms values rather than peak values. For instance, the 110 V available at every household is the rms value of the voltage from the power company. It is convenient in power analysis to express voltage and current in their rms values. Also, analog voltmeters and ammeters are designed to read directly the rms value of voltage and current, respectively.

EXAMPLE 11.7

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.

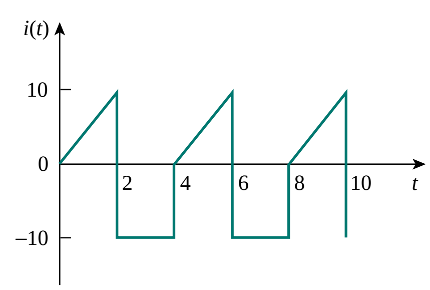


Figure 11.14 For Example 11.7.

Solution:

The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$\begin{aligned} I_{\text{rms}} &= \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]} \\ &= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\Big|_0^2 + 100t\Big|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165 \text{ A} \end{aligned}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3 \text{ W}$$

PRACTICE PROBLEM 11.7

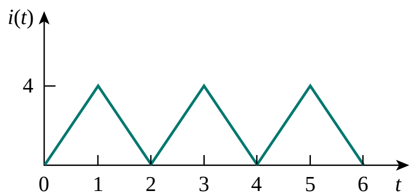


Figure 11.15 For Practice Prob. 11.7.

Find the rms value of the current waveform of Fig. 11.15. If the current flows through a 9-Ω resistor, calculate the average power absorbed by the resistor.

Answer: 2.309 A, 48 W.

EXAMPLE 11.8

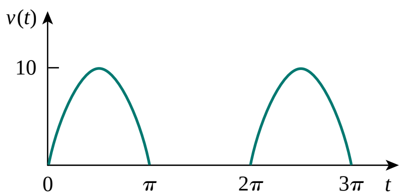


Figure 11.16 For Example 11.8.

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

Solution:

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^\pi (10\sin t)^2\,dt + \int_\pi^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^\pi \frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^\pi$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5\text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

PRACTICE PROBLEM 11.8

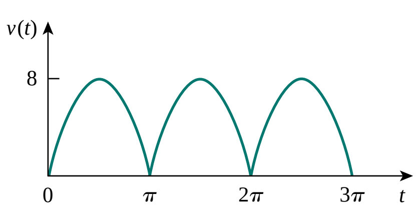


Figure 11.17 For Practice Prob. 11.8.

Find the rms value of the full-wave rectified sine wave in Fig. 11.17. Calculate the average power dissipated in a 6-Ω resistor.

Answer: 5.657 V, 5.334 W.

11.5 APPARENT POWER AND POWER FACTOR

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In Section 11.2 we see that if the voltage and current at the terminals of a circuit are

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad \text{and} \qquad i(t) = I_m \cos(\omega t + \theta_i) \tag{11.32}$$

or, in phasor form, $\mathbf{V} = V_m \underline{/\theta_v}$ and $\mathbf{I} = I_m \underline{/\theta_i}$, the average power is

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \tag{11.33}$$

In Section 11.4, we saw that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i) \tag{11.34}$$

We have added a new term to the equation:

$$\boxed{S = V_{\text{rms}} I_{\text{rms}}} \tag{11.35}$$

The average power is a product of two terms. The product $V_{\text{rms}} I_{\text{rms}}$ is known as the *apparent power* S. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

> The apparent power (in VA) is the product of the rms values of voltage and current.

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or real power, which is measured in watts. The power factor is dimensionless, since it is the ratio of the average power to the apparent power,

$$\boxed{\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)} \tag{11.36}$$

The angle $\theta_v - \theta_i$ is called the *power factor angle*, since it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if $\mathbf{V}$ is the voltage across the load and $\mathbf{I}$ is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \underline{/\theta_v}}{I_m \underline{/\theta_i}} = \frac{V_m}{I_m} \underline{/\theta_v - \theta_i} \tag{11.37}$$

Alternatively, since

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \underline{/\theta_v} \tag{11.38a}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \underline{/\theta_i} \tag{11.38b}$$

the impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \underline{/\theta_v - \theta_i} \tag{11.39}$$

The **power factor** is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

From Eq. (11.36), the power factor may also be regarded as the ratio of the real power dissipated in the load to the apparent power of the load.

From Eq. (11.36), the power factor may be seen as that factor by which the apparent power must be multiplied to obtain the real or average power. The value of pf ranges between zero and unity. For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf $= 1$. This implies that the apparent power is equal to the average power. For a purely reactive load, $\theta_v - \theta_i = \pm 90^\circ$ and pf $= 0$. In this case the average power is zero. In between these two extreme cases, pf is said to be *leading* or *lagging*. Leading power factor means that current leads voltage, which implies a capacitive load. Lagging power factor means that current lags voltage, implying an inductive load. Power factor affects the electric bills consumers pay the electric utility companies, as we will see in Section 11.9.2.

EXAMPLE 11.9

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\underline{/-20^\circ}}{4\underline{/10^\circ}} = 30\underline{/-30^\circ} = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30^\circ) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C}$$

or

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

PRACTICE PROBLEM 11.9

Obtain the power factor and the apparent power of a load whose impedance is $\mathbf{Z} = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 150\cos(377t + 10^\circ)$ V.

Answer: 0.832 lagging, 156 VA.

EXAMPLE 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

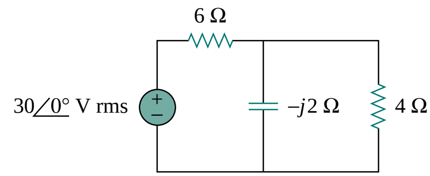

Figure 11.18 For Example 11.10.

Solution:

The total impedance is

$$\mathbf{Z} = 6 + 4 \parallel (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\underline{/-13.24}\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\underline{/0^\circ}}{7\underline{/-13.24^\circ}} = 4.286\underline{/13.24^\circ}\ \text{A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}}\ \text{pf} = (30)(4.286)0.9734 = 125\ \text{W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125\ \text{W}$$

where R is the resistive part of $\mathbf{Z}$.

PRACTICE PROBLEM 11.10

Calculate the power factor of the entire circuit of Fig. 11.19 as seen by the source. What is the average power supplied by the source?

Answer: 0.936 lagging, 118 W.

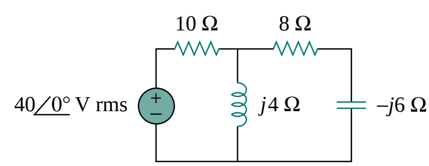

Figure 11.19 For Practice Prob. 11.10.

11.6 COMPLEX POWER

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Considerable effort has been expended over the years to express power relations as simply as possible. Power engineers have coined the term *complex power*, which they use to find the total effect of parallel loads. Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

Consider the ac load in Fig. 11.20. Given the phasor form $\mathbf{V} = V_m\underline{/\theta_v}$ and $\mathbf{I} = I_m\underline{/\theta_i}$ of voltage $v(t)$ and current $i(t)$, the *complex power* $\mathbf{S}$ absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* \tag{11.40}$$

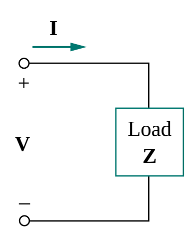

Figure 11.20 The voltage and current phasors associated with a load.

assuming the passive sign convention (see Fig. 11.20). In terms of the rms values,

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* \tag{11.41}$$

where

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \underline{/\theta_v} \tag{11.42}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \underline{/\theta_i} \tag{11.43}$$

When working with the rms values of currents or voltages, we may drop the subscript rms if no confusion will be caused by doing so.

Thus we may write Eq. (11.41) as

$$\begin{aligned} \mathbf{S} &= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \end{aligned} \tag{11.44}$$

This equation can also be obtained from Eq. (11.9). We notice from Eq. (11.44) that the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle.

The complex power may be expressed in terms of the load impedance **Z**. From Eq. (11.37), the load impedance **Z** may be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \underline{/\theta_v - \theta_i} \tag{11.45}$$

Thus, $\mathbf{V}_{\text{rms}} = \mathbf{Z}\mathbf{I}_{\text{rms}}$. Substituting this into Eq. (11.41) gives

$$\boxed{\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*}} \tag{11.46}$$

Since $\mathbf{Z} = R + jX$, Eq. (11.46) becomes

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ \tag{11.47}$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \tag{11.48}$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \tag{11.49}$$

P is the average or real power and it depends on the load's resistance R. Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

Comparing Eq. (11.44) with Eq. (11.47), we notice that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \tag{11.50}$$

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know from Chapter 6 that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source. Notice that:

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus,

> **Complex power** (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\boxed{\begin{aligned} \text{Complex Power} = \mathbf{S} = P + jQ &= \frac{1}{2}\mathbf{VI}^* \\ &= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} \\ \text{Apparent Power} = S = |\mathbf{S}| &= V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2} \\ \text{Real Power} = P &= \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\ \text{Reactive Power} = Q &= \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\ \text{Power Factor} &= \frac{P}{S} = \cos(\theta_v - \theta_i) \end{aligned}} \tag{11.51}$$

This shows how the complex power contains *all* the relevant power information in a given load.

S contains *all* power information of a load. The real part of **S** is the real power P; its imaginary part is the reactive power Q; its magnitude is the apparent power S; and the cosine of its phase angle is the power factor pf.

It is a standard practice to represent S, P, and Q in the form of a triangle, known as the *power triangle*, shown in Fig. 11.21(a). This is similar to the impedance triangle showing the relationship between $\mathbf{Z}$, R, and X, illustrated in Fig. 11.21(b). The power triangle has four items—the apparent/complex power, real power, reactive power, and the power factor angle. Given two of these items, the other two can easily be obtained from the triangle. As shown in Fig. 11.22, when **S** lies in the first quadrant, we have an inductive load and a lagging pf. When **S** lies in the fourth quadrant, the load is capacitive and the pf is leading. It is also possible for the complex power to lie in the second or third quadrant. This requires that the load impedance have a negative resistance, which is possible with active circuits.

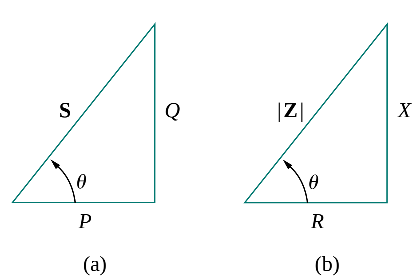

Figure 11.21 (a) Power triangle, (b) impedance triangle.

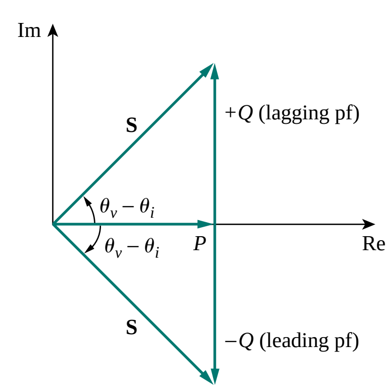

Figure 11.22 Power triangle.

EXAMPLE 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\underline{/-10^\circ}, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\underline{/+50^\circ}$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = \left(\frac{60}{\sqrt{2}}\underline{/-10^\circ}\right)\left(\frac{1.5}{\sqrt{2}}\underline{/-50^\circ}\right) = 45\underline{/-60^\circ} \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\underline{/-60^\circ} = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\underline{/-10^\circ}}{1.5\underline{/+50^\circ}} = 40\underline{/-60^\circ}\ \Omega$$

which is a capacitive impedance.

PRACTICE PROBLEM 11.11

For a load, $\mathbf{V}_{\text{rms}} = 110\underline{/85^\circ}$ V, $\mathbf{I}_{\text{rms}} = 0.4\underline{/15^\circ}$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Answer: (a) $44\underline{/70^\circ}$ VA, 44 VA, (b) 15.05 W, 41.35 VAR, (c) 0.342 lagging, $94.06 + j258.4\ \Omega$.

EXAMPLE 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf $= \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13°$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$, we obtain

$$\mathbf{I}^*_{\text{rms}} = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\underline{/0°}} = 85.6 + j51.7 \text{ A} = 100\underline{/31.13°} \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\underline{/-31.13°}$ and the peak current is

$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{120\underline{/0°}}{100\underline{/-31.13°}} = 1.2\underline{/31.13°}\ \Omega$$

which is an inductive impedance.

PRACTICE PROBLEM 11.12

A sinusoidal source supplies 10 kVA reactive power to load $\mathbf{Z} = 250\underline{/-75°}\ \Omega$. Determine: (a) the power factor, (b) the apparent power delivered to the load, and (c) the peak voltage.

Answer: (a) 0.2588 leading, (b) −10.35 kVAR, (c) 2.275 kV.

†11.7 CONSERVATION OF AC POWER

In fact, we already saw in Examples 11.3 and 11.4 that average power is conserved in ac circuits.

The principle of conservation of power applies to ac circuits as well as to dc circuits (see Section 1.5).

To see this, consider the circuit in Fig. 11.23(a), where two load impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are connected in parallel across an ac source $\mathbf{V}$. KCL gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 \tag{11.52}$$

The complex power supplied by the source is

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}\mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) = \frac{1}{2}\mathbf{V}\mathbf{I}_1^* + \frac{1}{2}\mathbf{V}\mathbf{I}_2^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.53}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

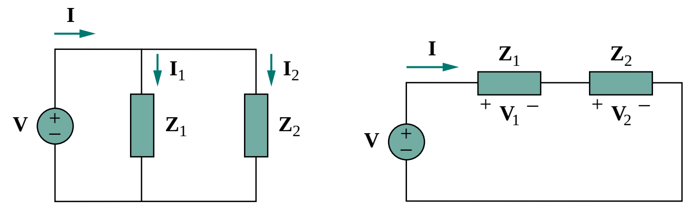

Figure 11.23 An ac voltage source supplied loads connected in: (a) parallel, (b) series.

If the loads are connected in series with the voltage source, as shown in Fig. 11.23(b), KVL yields

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{11.54}$$

The complex power supplied by the source is

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}(\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* = \frac{1}{2}\mathbf{V}_1\mathbf{I}^* + \frac{1}{2}\mathbf{V}_2\mathbf{I}^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.55}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

We conclude from Eqs. (11.53) and (11.55) that whether the loads are connected in series or in parallel (or in general), the total power *supplied* by the source equals the total power *delivered* to the load. Thus, in general, for a source connected to N loads,

$$\boxed{\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_1 + \cdots + \mathbf{S}_N} \tag{11.56}$$

This means that the total complex power in a network is the sum of the complex powers of the individual components. (This is also true of real power and reactive power, but not true of apparent power.) This expresses the principle of conservation of ac power:

In fact, all forms of ac power are conserved: instantaneous, real, reactive, and complex.

> The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

From this we imply that the real (or reactive) power flow from sources in a network equals the real (or reactive) power flow into the other elements in the network.

EXAMPLE 11.13

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

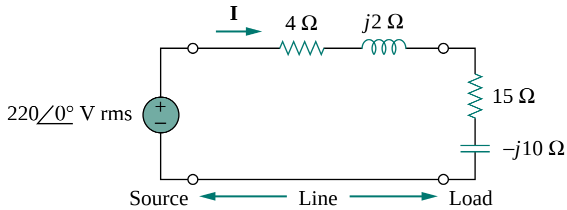

Figure 11.24 For Example 11.13.

Solution:

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\underline{/-22.83^\circ}\ \Omega$$

The current through the circuit is

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\underline{/0^\circ}}{20.62\underline{/-22.83^\circ}} = 10.67\underline{/22.83^\circ}\ \text{A rms}$$

(a) For the source, the complex power is

$$\begin{aligned}\mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* &= (220\underline{/0^\circ})(10.67\underline{/-22.83^\circ})\\ &= 2347.4\underline{/-22.83^\circ} = (2163.5 - j910.8)\ \text{VA}\end{aligned}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).
(b) For the line, the voltage is

$$\begin{aligned}\mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} &= (4.472\underline{/26.57^\circ})(10.67\underline{/22.83^\circ})\\ &= 47.72\underline{/49.4^\circ}\ \text{V rms}\end{aligned}$$

The complex power absorbed by the line is

$$\begin{aligned}\mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* &= (47.72\underline{/49.4^\circ})(10.67\underline{/-22.83^\circ})\\ &= 509.2\underline{/26.57^\circ} = 455.4 + j227.7\ \text{VA}\end{aligned}$$

or

$$\mathbf{S}_{\text{line}} = |\mathbf{I}|^2\mathbf{Z}_{\text{line}} = (10.67)^2(4 + j2) = 455.4 + j227.7\ \text{VA}$$

That is, the real power is 455.4 W and the reactive power is 227.76 VAR (lagging).
(c) For the load, the voltage is

$$\begin{aligned}\mathbf{V}_L = (15 - j10)\mathbf{I} &= (18.03\underline{/-33.7^\circ})(10.67\underline{/22.83^\circ})\\ &= 192.38\underline{/-10.87^\circ}\ \text{V rms}\end{aligned}$$

The complex power absorbed by the load is

$$\begin{aligned}\mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* &= (192.38\underline{/-10.87^\circ})(10.67\underline{/-22.83^\circ})\\ &= 2053\underline{/-33.7^\circ} = (1708 - j1139)\ \text{VA}\end{aligned}$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S}_s = \mathbf{S}_{\text{line}} + \mathbf{S}_L$, as expected. We have used the rms values of voltages and currents.

PRACTICE PROBLEM 11.13

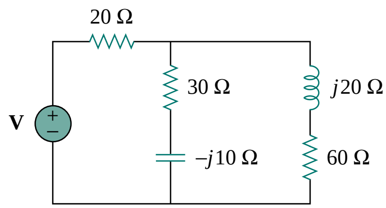

Figure 11.25 For Practice Prob. 11.13.

In the circuit in Fig. 11.25, the 60-Ω resistor absorbs an average power of 240 W. Find $\mathbf{V}$ and the complex power of each branch of the circuit. What is the overall complex power of the circuit?

Answer: $240.67\underline{/21.45^\circ}$ V (rms); the 20-Ω resistor: 656 VA; the $(30 - j10)$ Ω impedance: $480 - j160$ VA; the $(60 + j20)$ Ω impedance: $240 + j80$ VA; overall: $1376 - j80$ VA.

EXAMPLE 11.14

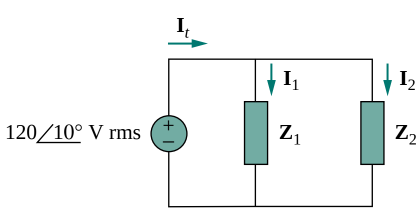

Figure 11.26 For Example 11.14.

In the circuit of Fig. 11.26, $\mathbf{Z}_1 = 60\underline{/-30^\circ}$ Ω and $\mathbf{Z}_2 = 40\underline{/45^\circ}$ Ω. Calculate the total: (a) apparent power, (b) real power, (c) reactive power, and (d) pf.

Solution:

The current through $\mathbf{Z}_1$ is

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{120\underline{/10^\circ}}{60\underline{/-30^\circ}} = 2\underline{/40^\circ} \text{ A rms}$$

while the current through $\mathbf{Z}_2$ is

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{120\underline{/10^\circ}}{40\underline{/45^\circ}} = 3\underline{/-35^\circ} \text{ A rms}$$

The complex powers absorbed by the impedances are

$$\mathbf{S}_1 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_1^*} = \frac{(120)^2}{60\underline{/30^\circ}} = 240\underline{/-30^\circ} = 207.85 - j120 \text{ VA}$$

$$\mathbf{S}_2 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_2^*} = \frac{(120)^2}{40\underline{/-45^\circ}} = 360\underline{/45^\circ} = 254.6 + j254.6 \text{ VA}$$

The total complex power is

$$\mathbf{S}_t = \mathbf{S}_1 + \mathbf{S}_2 = 462.4 + j134.6 \text{ VA}$$

(a) The total apparent power is
$|\mathbf{S}_t| = \sqrt{462.4^2 + 134.6^2} = 481.6$ VA.
(b) The total real power is
$P_t = \text{Re}(\mathbf{S}_t) = 462.4$ W or $P_t = P_1 + P_2$.
(c) The total reactive power is
$Q_t = \text{Im}(\mathbf{S}_t) = 134.6$ VAR or $Q_t = Q_1 + Q_2$.
(d) The pf $= P_t/|\mathbf{S}_t| = 462.4/481.6 = 0.96$ (lagging).

We may cross check the result by finding the complex power $\mathbf{S}_s$ supplied by the source.

$$\mathbf{I}_t = \mathbf{I}_1 + \mathbf{I}_2 = (1.532 + j1.286) + (2.457 - j1.721)$$
$$= 4 - j0.435 = 4.024\underline{/-6.21^\circ} \text{ A rms}$$
$$\mathbf{S}_s = \mathbf{V}\mathbf{I}_t^* = (120\underline{/10^\circ})(4.024\underline{/6.21^\circ})$$
$$= 482.88\underline{/16.21^\circ} = 463 + j135 \text{ VA}$$

which is the same as before.

PRACTICE PROBLEM 11.14

Two loads connected in parallel are respectively 2 kW at a pf of 0.75 leading and 4 kW at a pf of 0.95 lagging. Calculate the pf of the two loads. Find the complex power supplied by the source.

Answer: 0.9972 (leading), $6 - j0.4495$ kVA.

11.8 POWER FACTOR CORRECTION

Most domestic loads (such as washing machines, air conditioners, and refrigerators) and industrial loads (such as induction motors) are inductive and operate at a low lagging power factor. Although the inductive nature of the load cannot be changed, we can increase its power factor.

> The process of increasing the power factor without altering the voltage or current to the original load is known as power factor correction.

Alternatively, power factor correction may be viewed as the addition of a reactive element (usually a capacitor) in parallel with the load in order to make the power factor closer to unity.

Since most loads are inductive, as shown in Fig. 11.27(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. 11.27(b). The effect of adding the capacitor can be illustrated using either the power triangle or the phasor diagram of the currents involved. Figure 11.28 shows the latter, where it is assumed that the circuit in Fig. 11.27(a) has a power factor of $\cos\theta_1$, while the one in Fig. 11.27(b) has a power factor of $\cos\theta_2$. It is

An inductive load is modeled as a series combination of an inductor and a resistor.

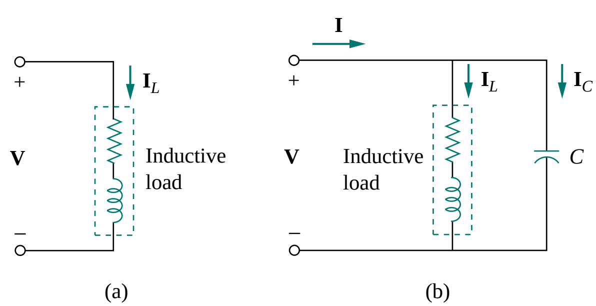

Figure 11.27 Power factor correction: (a) original inductive load, (b) inductive load with improved power factor.

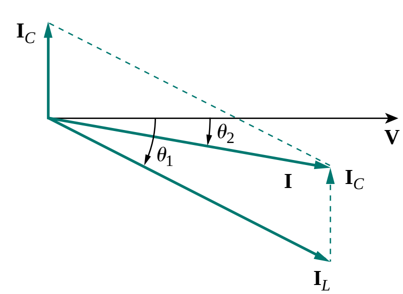

Figure 11.28 Phasor diagram showing the effect of adding a capacitor in parallel with the inductive load.

evident from Fig. 11.28 that adding the capacitor has caused the phase angle between the supplied voltage and current to reduce from θ_1 to θ_2, thereby increasing the power factor. We also notice from the magnitudes of the vectors in Fig. 11.28 that with the same supplied voltage, the circuit in Fig. 11.27(a) draws larger current I_L than the current I drawn by the circuit in Fig. 11.27(b). Power companies charge more for larger currents, because they result in increased power losses (by a squared factor, since $P = I_L^2 R$). Therefore, it is beneficial to both the power company and the consumer that every effort is made to minimize current level or keep the power factor as close to unity as possible. By choosing a suitable size for the capacitor, the current can be made to be completely in phase with the voltage, implying unity power factor.

We can look at the power factor correction from another perspective. Consider the power triangle in Fig. 11.29. If the original inductive load has apparent power S_1, then

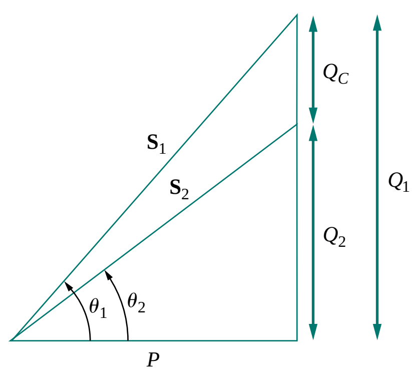

Figure 11.29 Power triangle illustrating power factor correction.

$$P = S_1 \cos\theta_1, \qquad Q_1 = S_1 \sin\theta_1 = P \tan\theta_1 \tag{11.57}$$

If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2 \cos\theta_2$), then the new reactive power is

$$Q_2 = P \tan\theta_2 \tag{11.58}$$

The reduction in the reactive power is caused by the shunt capacitor, that is,

$$Q_C = Q_1 - Q_2 = P(\tan\theta_1 - \tan\theta_2) \tag{11.59}$$

But from Eq. (11.49), $Q_C = V_{\text{rms}}^2 / X_C = \omega C V_{\text{rms}}^2$. The value of the required shunt capacitance C is determined as

$$\boxed{C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2}} \tag{11.60}$$

Note that the real power P dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

Although the most common situation in practice is that of an inductive load, it is also possible that the load is capacitive, that is, the load is operating at a leading power factor. In this case, an inductor should be connected across the load for power factor correction. The required shunt inductance L can be calculated from

$$Q_L = \frac{V_{\text{rms}}^2}{X_L} = \frac{V_{\text{rms}}^2}{\omega L} \qquad \Longrightarrow \qquad L = \frac{V_{\text{rms}}^2}{\omega Q_L} \tag{11.61}$$

where $Q_L = Q_1 - Q_2$, the difference between the new and old reactive powers.

EXAMPLE 11.15

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Solution:

If the pf $= 0.8$, then

$$\cos\theta_1 = 0.8 \qquad \Longrightarrow \qquad \theta_1 = 36.87^\circ$$

where θ_1 is the phase difference between voltage and current. We obtain the apparent power from the real power and the pf as

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000 \text{ VA}$$

The reactive power is

$$Q_1 = S_1 \sin\theta = 5000 \sin 36.87 = 3000 \text{ VAR}$$

When the pf is raised to 0.95,

$$\cos\theta_2 = 0.95 \qquad \Longrightarrow \qquad \theta_2 = 18.19^\circ$$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5 \text{ VA}$$

The new reactive power is

$$Q_2 = S_2 \sin\theta_2 = 1314.4 \text{ VAR}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

and

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$

@ Network Analysis

PRACTICE PROBLEM 11.15

Find the value of parallel capacitance needed to correct a load of 140 kVAR at 0.85 lagging pf to unity pf. Assume that the load is supplied by a 110-V (rms), 60-Hz line.

Answer: 30.69 mF.

†11.9 APPLICATIONS

In this section, we consider two important application areas: how power is measured and how electric utility companies determine the cost of electricity consumption.

Reactive power is measured by an instrument called the *varmeter*. The varmeter is often connected to the load in the same way as the wattmeter.

11.9.1 Power Measurement

The average power absorbed by a load is measured by an instrument called the *wattmeter*.

The **wattmeter** is the instrument for measuring the average power.

Some wattmeters do not have coils; the wattmeter considered here is the electromagnetic type.

Figure 11.30 shows a wattmeter that consists essentially of two coils: the current coil and the voltage coil. A current coil with very low impedance (ideally zero) is connected in series with the load (Fig. 11.31) and responds to the load current. The voltage coil with very high impedance (ideally infinite) is connected in parallel with the load as shown in Fig. 11.31 and responds to the load voltage. The current coil acts like a short circuit because of its low impedance; the voltage coil behaves like an open circuit because of its high impedance. As a result, the presence of the wattmeter does not disturb the circuit or have an effect on the power measurement.

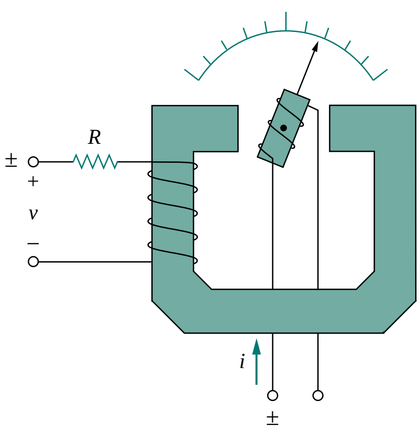

Figure 11.30 A wattmeter.

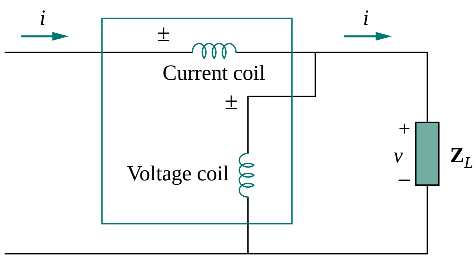

Figure 11.31 The wattmeter connected to the load.

When the two coils are energized, the mechanical inertia of the moving system produces a deflection angle that is proportional to the average value of the product $v(t)i(t)$. If the current and voltage of the load are $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, their corresponding rms phasors are

$$\mathbf{V}_{\text{rms}} = \frac{V_m}{\sqrt{2}} \angle \theta_v \qquad \text{and} \qquad \mathbf{I}_{\text{rms}} = \frac{I_m}{\sqrt{2}} \angle \theta_i \tag{11.62}$$

and the wattmeter measures the average power given by

$$P = |\mathbf{V}_{\text{rms}}||\mathbf{I}_{\text{rms}}| \cos(\theta_v - \theta_i) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \tag{11.63}$$

As shown in Fig. 11.31, each wattmeter coil has two terminals with one marked ±. To ensure upscale deflection, the ± terminal of the current coil is toward the source, while the ± terminal of the voltage coil is connected to the same line as the current coil. Reversing both coil connections still results in upscale deflection. However, reversing one coil and not the other results in downscale deflection and no wattmeter reading.

EXAMPLE 11.16

Find the wattmeter reading of the circuit in Fig. 11.32.

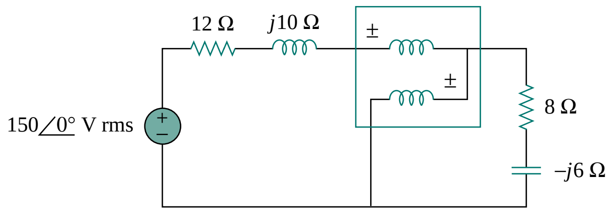

Figure 11.32 For Example 11.16.

Solution:

In Fig. 11.32, the wattmeter reads the average power absorbed by the $(8 - j6)$ Ω impedance because the current coil is in series with the impedance while the voltage coil is in parallel with it. The current through the circuit is

$$\mathbf{I} = \frac{150\underline{/0^\circ}}{(12 + j10) + (8 - j6)} = \frac{150}{20 + j4} \text{ A rms}$$

The voltage across the $(8 - j6)$ Ω impedance is

$$\mathbf{V} = \mathbf{I}(8 - j6) = \frac{150(8 - j6)}{20 + j4} \text{ V rms}$$

The complex power is

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = \frac{150(8 - j6)}{20 + j4} \cdot \frac{150}{20 - j4} = \frac{150^2(8 - j6)}{20^2 + 4^2}$$

$$= 423.7 - j324.6 \text{ VA}$$

The wattmeter reads

$$P = \text{Re}(\mathbf{S}) = 432.7 \text{ W}$$

PRACTICE PROBLEM 11.16

For the circuit in Fig. 11.33, find the wattmeter reading.

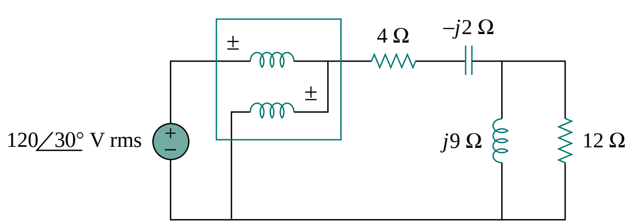

Figure 11.33 For Practice Prob. 11.16.

Answer: 1437 W.

11.9.2 Electricity Consumption Cost

In Section 1.7, we considered a simplified model of the way the cost of electricity consumption is determined. But the concept of power factor was not included in the calculations. Now we consider the importance of power factor in electricity consumption cost.

Loads with low power factors are costly to serve because they require large currents, as explained in Section 11.8. The ideal situation would be to draw minimum current from a supply so that $S = P$, $Q = 0$, and pf $= 1$. A load with nonzero Q means that energy flows forth and back between the load and the source, giving rise to additional power losses. In view of this, power companies often encourage their customers to have power factors as close to unity as possible and penalize some customers who do not improve their load power factors.

Utility companies divide their customers into categories: as residential (domestic), commercial, and industrial, or as small power, medium power, and large power. They have different rate structures for each category. The amount of energy consumed in units of kilowatt-hours (kWh) is measured using a kilowatt-hour meter installed at the customer's premises.

Although utility companies use different methods for charging customers, the tariff or charge to a consumer is often two-part. The first part is fixed and corresponds to the cost of generation, transmission, and distribution of electricity to meet the load requirements of the consumers. This part of the tariff is generally expressed as a certain price per kW of maximum demand. Or it may instead be based on kVA of maximum demand, to account for the power factor (pf) of the consumer. A pf penalty charge may be imposed on the consumer whereby a certain percentage of kW or kVA maximum demand is charged for every 0.01 fall in pf below a prescribed value, say 0.85 or 0.9. On the other hand, a pf credit may be given for every 0.01 that the pf exceeds the prescribed value.

The second part is proportional to the energy consumed in kWh; it may be in graded form, for example, the first 100 kWh at 16 cents/kWh, the next 200 kWh at 10 cents/kWh and so forth. Thus, the bill is determined based on the following equation:

$$\text{Total Cost} = \text{Fixed Cost} + \text{Cost of Energy} \tag{11.64}$$

EXAMPLE 11.17

A manufacturing industry consumes 200 MWh in one month. If the maximum demand is 1600 kW, calculate the electricity bill based on the following two-part rate:

Demand charge: \$5.00 per month per kW of billing demand.

Energy charge: 8 cents per kWh for the first 50,000 kWh, 5 cents per kWh for the remaining energy.

Solution:

The demand charge is

$$\$5.00 \times 1600 = \$8000 \tag{11.17.1}$$

The energy charge for the first 50,000 kWh is

$$\$0.08 \times 50{,}000 = \$4000 \tag{11.17.2}$$

The remaining energy is 200,000 kWh − 50,000 kWh = 150,000 kWh, and the corresponding energy charge is

$$\$0.05 \times 150{,}000 = \$7500 \tag{11.17.3}$$

Adding Eqs. (11.17.1) to (11.17.3) gives

$$\text{Total bill for the month} = \$8000 + \$4000 + \$7500 = \$19{,}500$$

It may appear that the cost of electricity is too high. But this is often a small fraction of the overall cost of production of the goods manufactured or the selling price of the finished product.

PRACTICE PROBLEM 11.17

The monthly reading of a paper mill's meter is as follows:

Maximum demand: 32,000 kW

Energy consumed: 500 MWh

Using the two-part rate in Example 11.17, calculate the monthly bill for the paper mill.

Answer: $186,500.

EXAMPLE 11.18

A 300-kW load supplied at 13 kV (rms) operates 520 hours a month at 80 percent power factor. Calculate the average cost per month based on this simplified tariff:

Energy charge: 6 cents per kWh

Power-factor penalty: 0.1 percent of energy charge for every 0.01 that pf falls below 0.85.

Power-factor credit: 0.1 percent of energy charge for every 0.01 that pf exceeds 0.85.

Solution:

The energy consumed is

$$W = 300 \text{ kW} \times 520 \text{ h} = 156{,}000 \text{ kWh}$$

The operating power factor pf = 80% = 0.8 is 5 × 0.01 below the prescribed power factor of 0.85. Since there is 0.1 percent energy charge for every 0.01, there is a power-factor penalty charge of 0.5 percent. This amounts to an energy charge of

$$\Delta W = 156{,}000 \times \frac{5 \times 0.1}{100} = 780 \text{ kWh}$$

The total energy is

$$W_t = W + \Delta W = 156{,}000 + 780 = 156{,}780 \text{ kWh}$$

The cost per month is given by

$$\text{Cost} = 6 \text{ cents} \times W_t = \$0.06 \times 156{,}780 = \$9406.80$$

PRACTICE PROBLEM 11.18

An 800-kW induction furnace at 0.88 power factor operates 20 hours per day for 26 days in a month. Determine the electricity bill per month based on the tariff in Example 11.16.

Answer: $24,885.12.

11.10 SUMMARY

1. The instantaneous power absorbed by an element is the product of the element's terminal voltage and the current through the element: $p = vi$.
2. Average or real power P (in watts) is the average of instantaneous power p:

$$P = \frac{1}{T}\int_0^T p\,dt$$

If $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, then $V_{\text{rms}} = V_m/\sqrt{2}$, $I_{\text{rms}} = I_m/\sqrt{2}$, and

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

Inductors and capacitors absorb no average power, while the average power absorbed by a resistor is $1/2\, I_m^2 R = I_{\text{rms}}^2 R$.
3. Maximum average power is transferred to a load when the load impedance is the complex conjugate of the Thevenin impedance as seen from the load terminals, $\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^*$.
4. The effective value of a periodic signal $x(t)$ is its root-mean-square (rms) value.

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

For a sinusoid, the effective or rms value is its amplitude divided by $\sqrt{2}$.
5. The power factor is the cosine of the phase difference between voltage and current:

$$\text{pf} = \cos(\theta_v - \theta_i)$$

It is also the cosine of the angle of the load impedance or the ratio of real power to apparent power. The pf is lagging if the current lags voltage (inductive load) and is leading when the current leads voltage (capacitive load).

6. Apparent power S (in VA) is the product of the rms values of voltage and current:

$$S = V_{\text{rms}} I_{\text{rms}}$$

It is also given by $S = |\mathbf{S}| = \sqrt{P^2 + Q^2}$, where Q is reactive power.

7. Reactive power (in VAR) is:

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

8. Complex power $\mathbf{S}$ (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. It is also the complex sum of real power P and reactive power Q.

$$\mathbf{S} = \mathbf{V}_{\text{rms}} \mathbf{I}^*_{\text{rms}} = V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} = P + jQ$$

Also,

$$\mathbf{S} = I^2_{\text{rms}} \mathbf{Z} = \frac{V^2_{\text{rms}}}{\mathbf{Z}^*}$$

9. The total complex power in a network is the sum of the complex powers of the individual components. Total real power and reactive power are also, respectively, the sums of the individual real powers and the reactive powers, but the total apparent power is not calculated by the process.

10. Power factor correction is necessary for economic reasons; it is the process of improving the power factor of a load by reducing the overall reactive power.

11. The wattmeter is the instrument for measuring the average power. Energy consumed is measured with a kilowatt-hour meter.

REVIEW QUESTIONS

11.1 The average power absorbed by an inductor is zero.
(a) True (b) False

11.2 The Thevenin impedance of a network seen from the load terminals is $80 + j55\ \Omega$. For maximum power transfer, the load impedance must be:
(a) $-80 + j55\ \Omega$ (b) $-80 - j55\ \Omega$
(c) $80 - j55\ \Omega$ (d) $80 + j55\ \Omega$

11.3 The amplitude of the voltage available in the 60-Hz, 120-V power outlet in your home is:
(a) 110 V (b) 120 V
(c) 170 V (d) 210 V

11.4 If the load impedance is $20 - j20$, the power factor is
(a) $\underline{/-45^\circ}$ (b) 0 (c) 1
(d) 0.7071 (e) none of these

11.5 A quantity that contains all the power information in a given load is the
(a) power factor (b) apparent power
(c) average power (d) reactive power
(e) complex power

11.6 Reactive power is measured in:
(a) watts (b) VA
(c) VAR (d) none of these

11.7 In the power triangle shown in Fig. 11.34(a), the reactive power is:
(a) 1000 VAR leading (b) 1000 VAR lagging
(c) 866 VAR leading (d) 866 VAR lagging

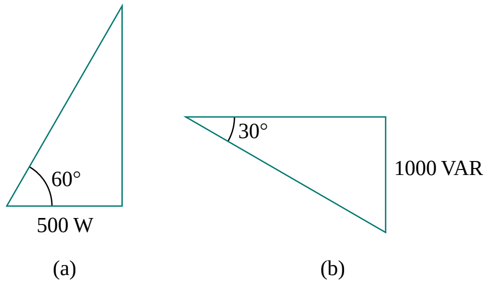

Figure 11.34 For Review Questions 11.7 and 11.8.

11.8 For the power triangle in Fig. 11.34(b), the apparent power is:

(a) 2000 VA (b) 1000 VAR
(c) 866 VAR (d) 500 VAR

11.9 A source is connected to three loads $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ in parallel. Which of these is not true?

(a) $P = P_1 + P_2 + P_3$ (b) $Q = Q_1 + Q_2 + Q_3$
(c) $S = S_1 + S_2 + S_3$ (d) $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$

11.10 The instrument for measuring average power is the:

(a) voltmeter (b) ammeter
(c) wattmeter (d) varmeter
(e) kilowatt-hour meter

Answers: 11.1a, 11.2c, 11.3c, 11.4d, 11.5e, 11.6c, 11.7d, 11.8a, 11.9c, 11.10c.

PROBLEMS

Section 11.2 Instantaneous and Average Power

11.1 If $v(t) = 160 \cos 50t$ V and $i(t) = -20 \sin(50t - 30°)$ A, calculate the instantaneous power and the average power.

11.2 At $t = 2$ s, find the instantaneous power on each of the elements in the circuit of Fig. 11.35.

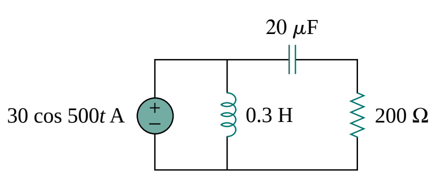

Figure 11.35 For Prob. 11.2.

11.3 Refer to the circuit depicted in Fig. 11.36. Find the average power absorbed by each element.

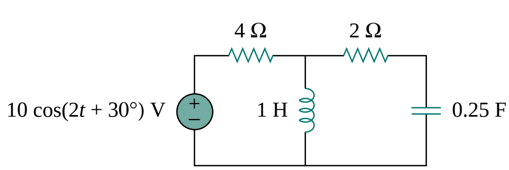

Figure 11.36 For Prob. 11.3.

11.4 Given the circuit in Fig. 11.37, find the average power absorbed by each of the elements.

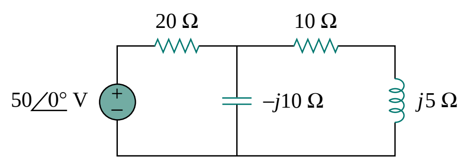

Figure 11.37 For Prob. 11.4.

11.5 Compute the average power absorbed by the 4-Ω resistor in the circuit of Fig. 11.38.

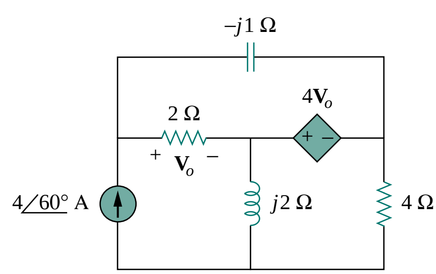

Figure 11.38 For Prob. 11.5.

11.6 Given the circuit of Fig. 11.39, find the average power absorbed by the 10-Ω resistor.

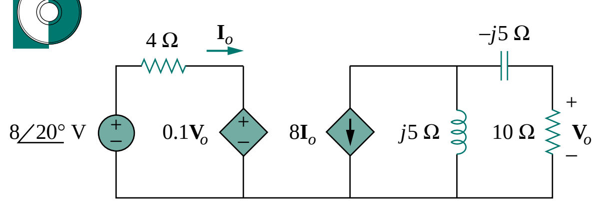

Figure 11.39 For Prob. 11.6.

11.7 In the circuit of Fig. 11.40, determine the average power absorbed by the 40-Ω resistor.

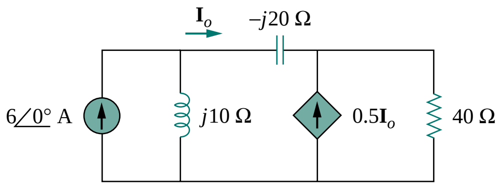

Figure 11.40 For Prob. 11.7.

11.8 Calculate the average power absorbed by each resistor in the op amp circuit of Fig. 11.41 if the rms value of v_s is 2 V.

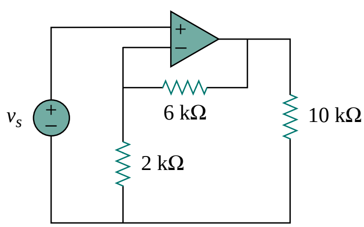

Figure 11.41 For Prob. 11.8.

11.9 In the op amp circuit in Fig. 11.42, find the total average power absorbed by the resistors.

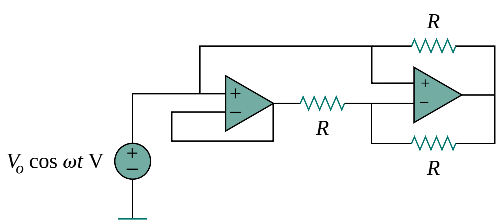

Figure 11.42 For Prob. 11.9.

11.10 For the network in Fig. 11.43, assume that the port impedance is

$$\mathbf{Z}_{ab} = \frac{R}{\sqrt{1 + \omega^2 R^2 C^2}} \angle -\tan^{-1} \omega RC$$

Find the average power consumed by the network when $R = 10\ \text{k}\Omega$, $C = 200$ nF, and $i = 2\sin(377t + 22°)$ mA.

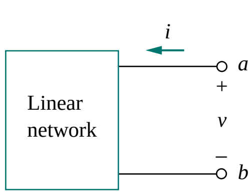

Figure 11.43 For Prob. 11.10.

Section 11.3 Maximum Average Power Transfer

11.11 For each of the circuits in Fig. 11.44, determine the value of load **Z** for maximum power transfer and the maximum average power transferred.

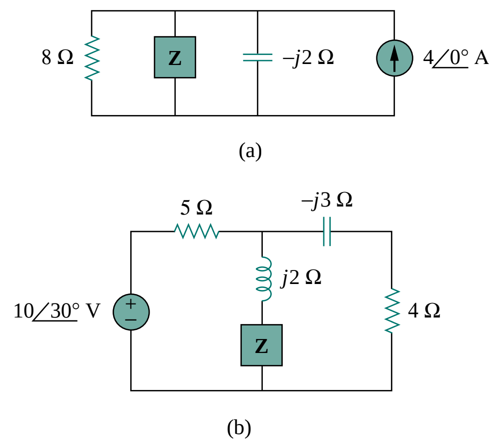

Figure 11.44 For Prob. 11.11.

11.12 For the circuit in Fig. 11.45, find:

(a) the value of the load impedance that absorbs the maximum average power

(b) the value of the maximum average power absorbed

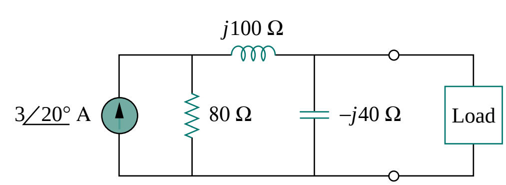

Figure 11.45 For Prob. 11.12.

11.13 In the circuit of Fig. 11.46, find the value of $\mathbf{Z}_L$ that will absorb the maximum power and the value of the maximum power.

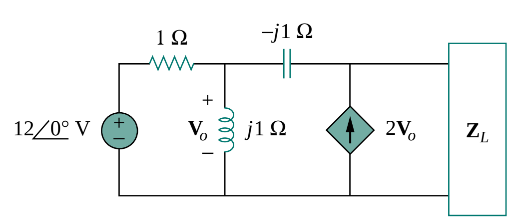

Figure 11.46 For Prob. 11.13.

11.14 Calculate the value of $\mathbf{Z}_L$ in the circuit of Fig. 11.47 in order for $\mathbf{Z}_L$ to receive maximum average power. What is the maximum average power received by $\mathbf{Z}$?

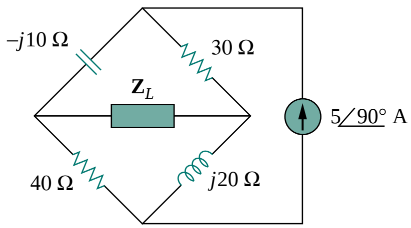

Figure 11.47 For Prob. 11.14.

11.15 Find the value of $\mathbf{Z}_L$ in the circuit of Fig. 11.48 for maximum power transfer.

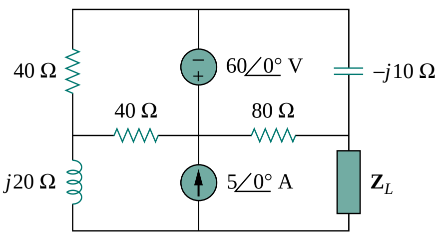

Figure 11.48 For Prob. 11.15.

11.16 The variable resistor R in the circuit of Fig. 11.49 is adjusted until it absorbs the maximum average power. Find R and the maximum average power absorbed.

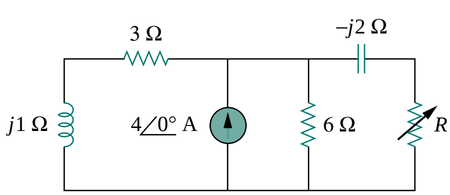

Figure 11.49 For Prob. 11.16.

11.17 The load resistance R_L in Fig. 11.50 is adjusted until it absorbs the maximum average power. Calculate the value of R_L and the maximum average power.

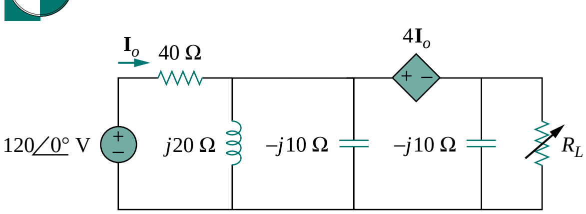

Figure 11.50 For Prob. 11.17.

11.18 Assuming that the load impedance is to be purely resistive, what load should be connected to terminals a-b of the circuits in Fig. 11.51 so that the maximum power is transferred to the load?

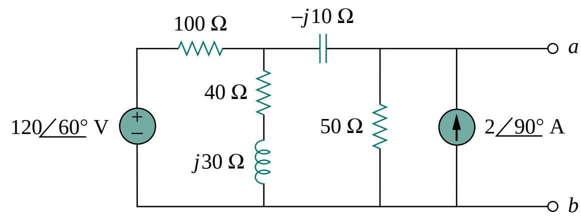

Figure 11.51 For Prob. 11.18.

Section 11.4 Effective or RMS Value

11.19 Find the rms value of the periodic signal in Fig. 11.52.

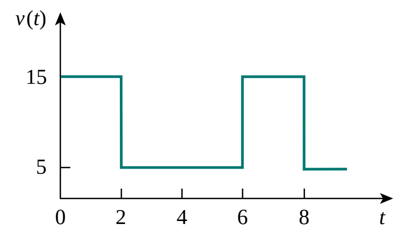

Figure 11.52 For Prob. 11.19.

11.20 Determine the rms value of the waveform in Fig. 11.53.

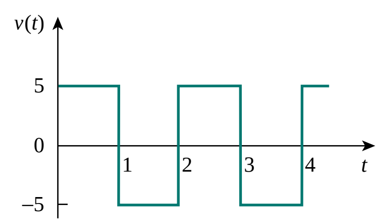

Figure 11.53 For Prob. 11.20.

11.21 Find the effective value of the voltage waveform in Fig. 11.54.

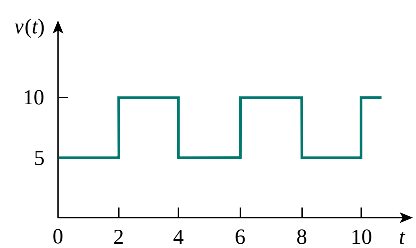

Figure 11.54 For Prob. 11.21.

11.22 Calculate the rms value of the current waveform of Fig. 11.55.

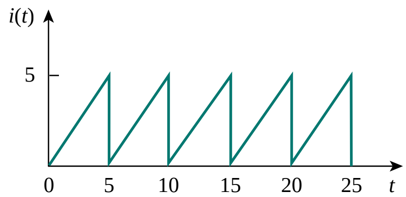

Figure 11.55 For Prob. 11.22.

11.23 Find the rms value of the voltage waveform of Fig. 11.56 as well as the average power absorbed by a 2-Ω resistor when the voltage is applied across the resistor.

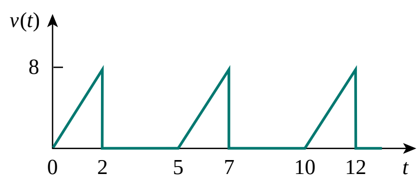

Figure 11.56 For Prob. 11.23.

11.24 Calculate the effective value of the current waveform in Fig. 11.57 and the average power delivered to a 12-Ω resistor when the current runs through the resistor.

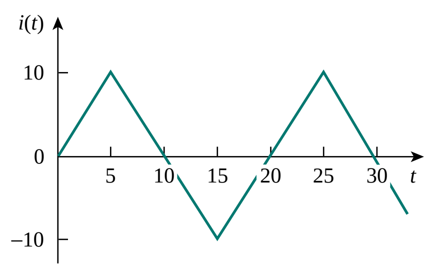

Figure 11.57 For Prob. 11.24.

11.25 Compute the rms value of the waveform depicted in Fig. 11.58.

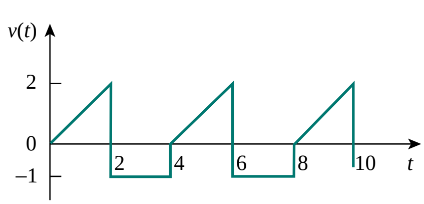

Figure 11.58 For Prob. 11.25.

11.26 Obtain the rms value of the current waveform shown in Fig. 11.59.

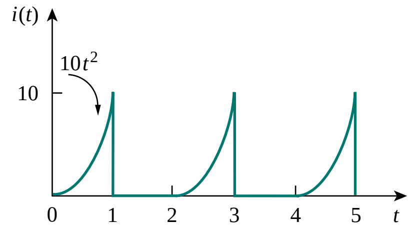

Figure 11.59 For Prob. 11.26.

11.27 Determine the effective value of the periodic waveform in Fig. 11.60.

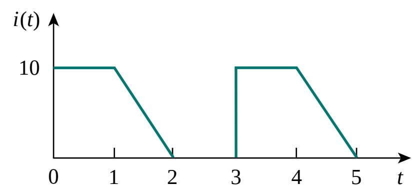

Figure 11.60 For Prob. 11.27.

11.28 One cycle of a periodic voltage waveform is depicted in Fig. 11.61. Find the effective value of the voltage.

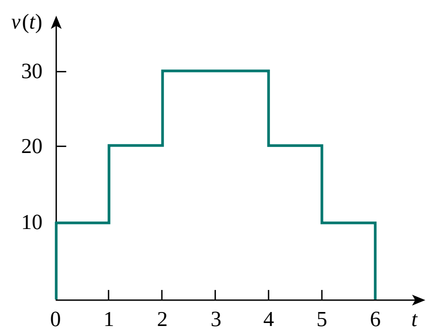

Figure 11.61 For Prob. 11.28.

Section 11.5 Apparent Power and Power Factor

11.29 A relay coil is connected to a 210-V, 50-Hz supply. If it has a resistance of 30 Ω and an inductance of 0.5 H, calculate the apparent power and the power factor.

11.30 A certain load comprises $12 - j8\ \Omega$ in parallel with $j4\ \Omega$. Determine the overall power factor.

11.31 Obtain the power factor for each of the circuits in Fig. 11.62. Specify each power factor as leading or lagging.

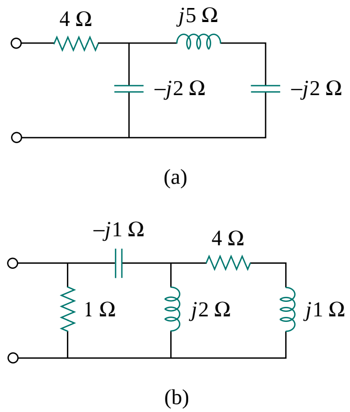

Figure 11.62 For Prob. 11.31.

Section 11.6 Complex Power

11.32 A load draws 5 kVAR at a power factor of 0.86 (leading) from a 220-V rms source. Calculate the peak current and the apparent power supplied to the load.

11.33 For the following voltage and current phasors, calculate the complex power, apparent power, real power, and reactive power. Specify whether the pf is leading or lagging.

(a) $\mathbf{V} = 220\angle 30^\circ$ V rms, $\mathbf{I} = 0.5\angle 60^\circ$ A rms

(b) $\mathbf{V} = 250\angle -10^\circ$ V rms,
$\mathbf{I} = 6.2\angle -25^\circ$ A rms

(c) $\mathbf{V} = 120\angle 0^\circ$ V rms, $\mathbf{I} = 2.4\angle -15^\circ$ A rms

(d) $\mathbf{V} = 160\angle 45^\circ$ V rms, $\mathbf{I} = 8.5\angle 90^\circ$ A rms

11.34 For each of the following cases, find the complex power, the average power, and the reactive power:

(a) $v(t) = 112\cos(\omega t + 10^\circ)$ V,
$i(t) = 4\cos(\omega t - 50^\circ)$ A

(b) $v(t) = 160\cos 377t$ V,
$i(t) = 4\cos(377t + 45^\circ)$ A

(c) $\mathbf{V} = 80\angle 60^\circ$ V rms, $\mathbf{Z} = 50\angle 30^\circ$ Ω

(d) $\mathbf{I} = 10\angle 60^\circ$ V rms, $\mathbf{Z} = 100\angle 45^\circ$ Ω

11.35 Determine the complex power for the following cases:

(a) $P = 269$ W, $Q = 150$ VAR (capacitive)

(b) $Q = 2000$ VAR, pf $= 0.9$ (leading)

(c) $S = 600$ VA, $Q = 450$ VAR (inductive)

(d) $V_{\text{rms}} = 220$ V, $P = 1$ kW,
$|\mathbf{Z}| = 40$ Ω (inductive)

11.36 Find the complex power for the following cases:

(a) $P = 4$ kW, pf $= 0.86$ (lagging)

(b) $S = 2$ kVA, $P = 1.6$ kW (capacitive)

(c) $\mathbf{V}_{\text{rms}} = 208\angle 20^\circ$ V, $\mathbf{I}_{\text{rms}} = 6.5\angle -50^\circ$ A

(d) $\mathbf{V}_{\text{rms}} = 120\angle 30^\circ$ V, $\mathbf{Z} = 40 + j60$ Ω

11.37 Obtain the overall impedance for the following cases:

(a) $P = 1000$ W, pf $= 0.8$ (leading),
$V_{\text{rms}} = 220$ V

(b) $P = 1500$ W, $Q = 2000$ VAR (inductive),
$I_{\text{rms}} = 12$ A

(c) $\mathbf{S} = 4500\angle 60^\circ$ VA, $\mathbf{V} = 120\angle 45^\circ$ V

11.38 For the entire circuit in Fig. 11.63, calculate:

(a) the power factor

(b) the average power delivered by the source

(c) the reactive power

(d) the apparent power

(e) the complex power

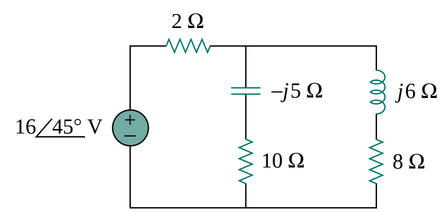

Figure 11.63 For Prob. 11.38.

Section 11.7 Conservation of AC Power

11.39 For the network in Fig. 11.64, find the complex power absorbed by each element.

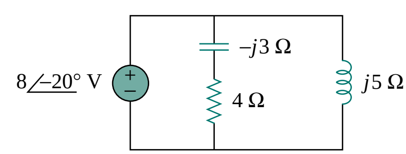

Figure 11.64 For Prob. 11.39.

11.40 Find the complex power absorbed by each of the five elements in the circuit of Fig. 11.65.

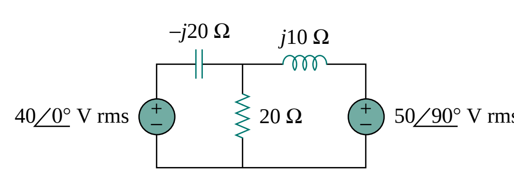

Figure 11.65 For Prob. 11.40.

11.41 Obtain the complex power delivered by the source in the circuit of Fig. 11.66.

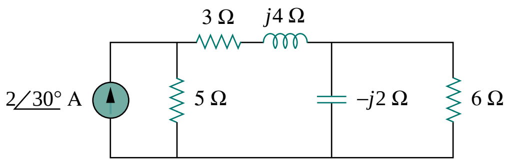

Figure 11.66 For Prob. 11.41.

11.42 For the circuit in Fig. 11.67, find the average, reactive, and complex power delivered by the dependent voltage source.

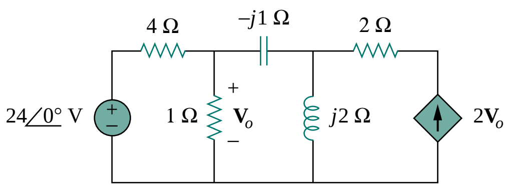

Figure 11.67 For Prob. 11.42.

11.43 Obtain the complex power delivered to the 10-kΩ resistor in Fig. 11.68 below.

11.44 Calculate the reactive power in the inductor and capacitor in the circuit of Fig. 11.69.

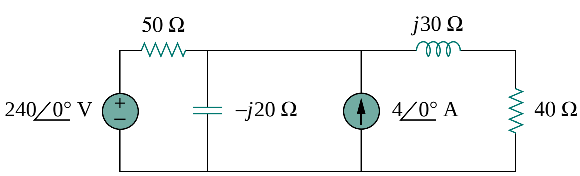

Figure 11.69 For Prob. 11.44.

11.45 For the circuit in Fig. 11.70, find $\mathbf{V}_o$ and the input power factor.

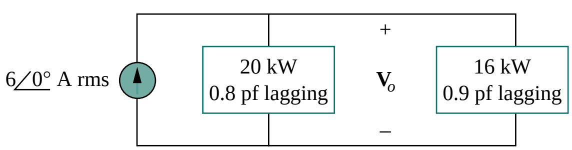

Figure 11.70 For Prob. 11.45.

11.46 Given the circuit in Fig. 11.71, find $\mathbf{I}_o$ and the overall complex power supplied.

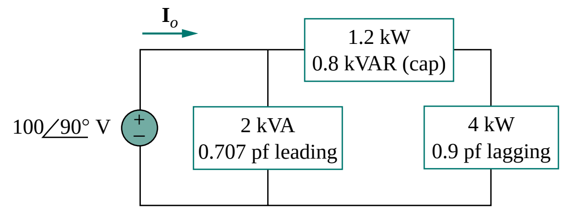

Figure 11.71 For Prob. 11.46.

11.47 For the circuit in Fig. 11.72, find $\mathbf{V}_s$.

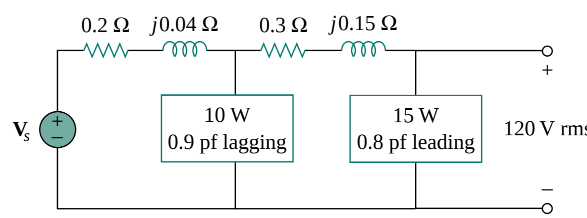

Figure 11.72 For Prob. 11.47.

11.48 Find $\mathbf{I}_o$ in the circuit of Fig. 11.73 on the bottom of the next page.

11.49 In the op amp circuit of Fig. 11.74, $v_s = 4\cos 10^4 t$ V. Find the average power delivered to the 50-kΩ resistor.

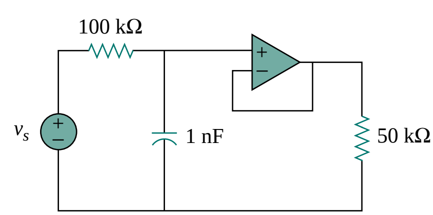

Figure 11.74 For Prob. 11.49.

11.50 Obtain the average power absorbed by the 6-kΩ resistor in the op amp circuit in Fig. 11.75.

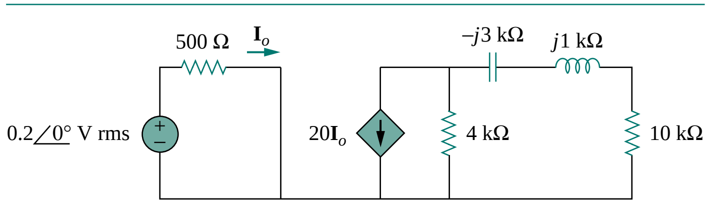

Figure 11.68 For Prob. 11.43.

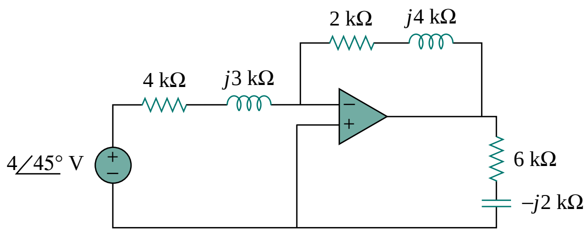

Figure 11.75 For Prob. 11.50.

11.51 Calculate the complex power delivered to each resistor and capacitor in the op amp circuit of Fig. 11.76. Let $v_s = 2 \cos 10^3 t$ V.

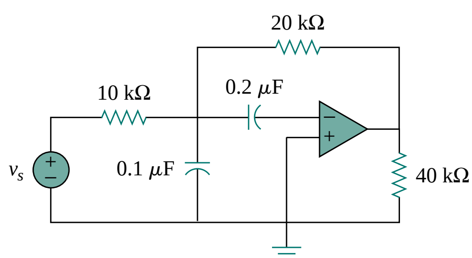

Figure 11.76 For Prob. 11.51.

11.52 Compute the complex power supplied by the current source in the series RLC circuit in Fig. 11.77.

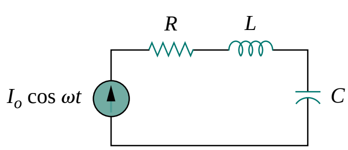

Figure 11.77 For Prob. 11.52.

Section 11.8 Power Factor Correction

11.53 Refer to the circuit shown in Fig. 11.78.

(a) What is the power factor?

(b) What is the average power dissipated?

(c) What is the value of the capacitance that will give a unity power factor when connected to the load?

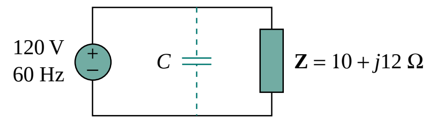

Figure 11.78 For Prob. 11.53.

11.54 An 880-VA, 220-V, 50-Hz load has a power factor of 0.8 lagging. What value of parallel capacitance will correct the load power factor to unity?

11.55 An 40-kW induction motor, with a lagging power factor of 0.76, is supplied by a 120-V rms 60-Hz sinusoidal voltage source. Find the capacitance needed in parallel with the motor to raise the power factor to:

(a) 0.9 lagging (b) 1.0.

11.56 A 240-V rms 60-Hz supply serves a load that is 10 kW (resistive), 15 kVAR (capacitive), and 22 kVAR (inductive). Find:

(a) the apparent power

(b) the current drawn from the supply

(c) the kVAR rating and capacitance required to improve the power factor to 0.96 lagging

(d) the current drawn from the supply under the new power-factor conditions

11.57 A 120-V rms 60-Hz source supplies two loads connected in parallel, as shown in Fig. 11.79.

(a) Find the power factor of the parallel combination.

(b) Calculate the value of the capacitance connected in parallel that will raise the power factor to unity.

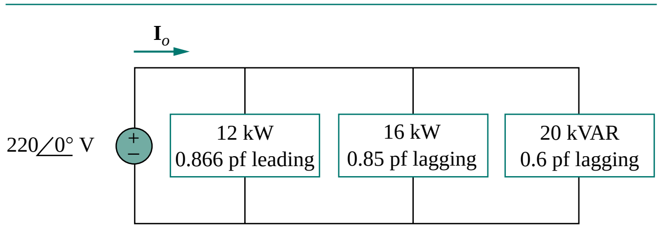

Figure 11.73 For Prob. 11.48.

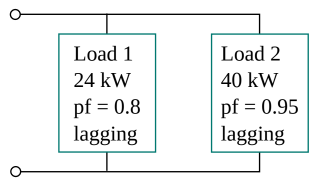

Figure 11.79 For Prob. 11.57.

11.58 Consider the power system shown in Fig. 11.80. Calculate:

(a) the total complex power

(b) the power factor

(c) the capacitance necessary to establish a unity power factor

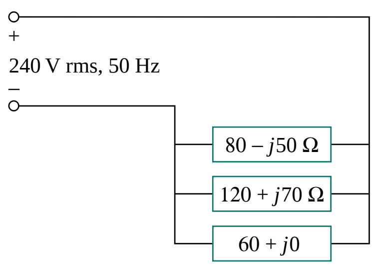

Figure 11.80 For Prob. 11.58.

Section 11.9 Applications

11.59 Obtain the wattmeter reading of the circuit in Fig. 11.81 below.

11.60 What is the reading of the wattmeter in the network of Fig. 11.82?

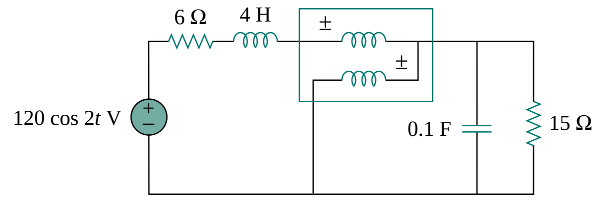

Figure 11.82 For Prob. 11.60.

11.61 Find the wattmeter reading of the circuit shown in Fig. 11.83 below.

11.62 The circuit of Fig. 11.84 portrays a wattmeter connected into an ac network.

(a) Find the load current.

(b) Calculate the wattmeter reading.

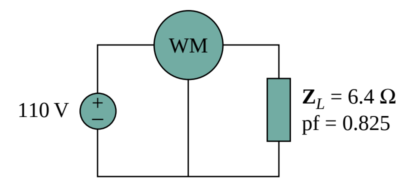

Figure 11.84 For Prob. 11.62.

11.63 The kilowatthour-meter of a home is read once a month. For a particular month, the previous and present readings are as follows:

Previous reading: 3246 kWh

Present reading: 4017 kWh

Calculate the electricity bill for that month based on the following residential rate schedule:

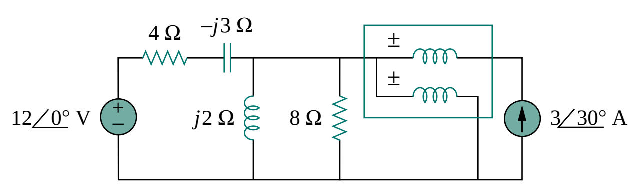

Figure 11.81 For Prob. 11.59.

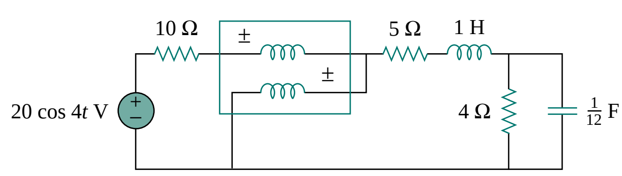

Figure 11.83 For Prob. 11.61.

Minimum monthly charge—\$12.00
First 100 kWh per month at 16 cents/kWh
Next 200 kWh per month at 10 cents/kWh
Over 300 kWh per month at 6 cents/kWh

11.64 A consumer has an annual consumption of 1200 MWh with a maximum demand of 2.4 MVA. The maximum demand charge is \$30 per kVA per annum, and the energy charge per kWh is 4 cents.

(a) Determine the annual cost of energy.

(b) Calculate the charge per kWh with a flat-rate tariff if the revenue to the utility company is to remain the same as for the two-part tariff.

COMPREHENSIVE PROBLEMS

11.65 A transmitter delivers maximum power to an antenna when the antenna is adjusted to represent a load of 75-Ω resistance in series with an inductance of 4 μH. If the transmitter operates at 4.12 MHz, find its internal impedance.

11.66 In a TV transmitter, a series circuit has an impedance of 3 kΩ and a total current of 50 mA. If the voltage across the resistor is 80 V, what is the power factor of the circuit?

11.67 A certain electronic circuit is connected to a 110-V ac line. The root-mean-square value of the current drawn is 2 A, with a phase angle of 55°.

(a) Find the true power drawn by the circuit.

(b) Calculate the apparent power.

11.68 An industrial heater has a nameplate which reads:

210 V 60 Hz 12 kVA 0.78 pf lagging

Determine:

(a) the apparent and the complex power

(b) the impedance of the heater

***11.69** A 2000-kW turbine-generator of 0.85 power factor operates at the rated load. An additional load of 300 kW at 0.8 power factor is added. What kVAR of capacitors is required to operate the turbine -generator but keep it from being overloaded?

11.70 The nameplate of an electric motor has the following information:

Line voltage: 220 V rms
Line current: 15 A rms
Line frequency: 60 Hz
Power: 2700 W

Determine the power factor (lagging) of the motor. Find the value of the capacitance C that must be connected across the motor to raise the pf to unity.

11.71 As shown in Fig. 11.85, a 550-V feeder line supplies an industrial plant consisting of a motor drawing 60 kW at 0.75 pf (inductive), a capacitor with a rating of 20 kVAR, and lighting drawing 20 kW.

(a) Calculate the total reactive power and apparent power absorbed by the plant.

(b) Determine the overall pf.

(c) Find the current in the feeder line.

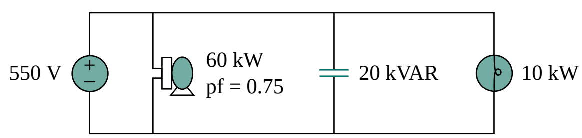

Figure 11.85 For Prob. 11.71.

11.72 A factory has the following four major loads:

- A motor rated at 5 hp, 0.8 pf lagging (1 hp = 0.7457 kW).
- A heater rated at 1.2 kW, 1.0 pf.
- Ten 120-W lightbulbs.
- A synchronous motor rated at 1.6 kVA, 0.6 pf leading.

(a) Calculate the total real and reactive power.

(b) Find the overall power factor.

11.73 A 1-MVA substation operates at full load at 0.7 power factor. It is desired to improve the power factor to 0.95 by installing capacitors. Assume that new substation and distribution facilities cost \$120 per kVA installed, and capacitors cost \$30 per kVA installed.

(a) Calculate the cost of capacitors needed.

(b) Find the savings in substation capacity released.

(c) Are capacitors economical for releasing the amount of substation capacity?

11.74 A coupling capacitor is used to block dc current from an amplifier as shown in Fig. 11.86(a). The amplifier and the capacitor act as the source, while the speaker is the load as in Fig. 11.86(b).

(a) At what frequency is maximum power transferred to the speaker?

(b) If $V_s = 4.6$ V rms, how much power is delivered to the speaker at that frequency?

*An asterisk indicates a challenging problem.

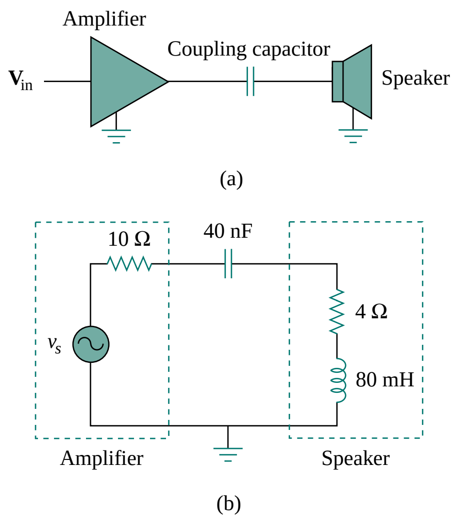

Figure 11.86 For Prob. 11.74.

11.75 A power amplifier has an output impedance of $40 + j8\ \Omega$. It produces a no-load output voltage of 146 V at 300 Hz.

(a) Determine the impedance of the load that achieves maximum power transfer.

(b) Calculate the load power under this matching condition.

11.76 A power transmission system is modeled as shown in Fig. 11.87. If $\mathbf{V}_s = 240\underline{/0^\circ}$ rms, find the average power absorbed by the load.

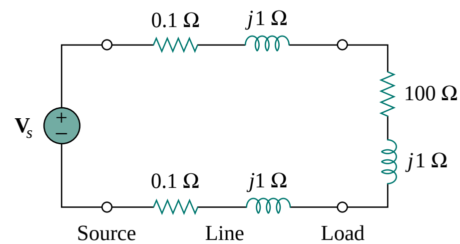

Figure 11.87 For Prob. 11.76.